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A Transformation for Symplectic Systems and the Definition of a Focal Point

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Abstract—We examine transformations for symplectic difference systems and Riccati difference operators connected with permutations of rows of a conjoined basis. The concept of an integration path for a conjoined basis is introduced to formulate the definition of a focal point and the disconjugacy criteria and state Sturm's separation theorems in terms of solutions of the transformed Riccati equation. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We consider transformations for the symplectic difference system

$$Y_{i+1} = W_i Y_i, \quad W_i^\top J_{2n} W_i = J_{2n}, \qquad i = 0, 1, \dots, N, \tag{1.1}$$

and the Riccati difference operator

$$R_W[Q] = \mathcal{C}_i - Q_{i+1}\mathcal{A}_i + \mathcal{D}_i Q_i - Q_{i+1}\mathcal{B}_i Q_i,$$

where W_i, Y_i, J_{2n} are real partitioned matrices with $n \times n$ blocks

$$W_i = \begin{bmatrix} \mathcal{A}_i & \mathcal{B}_i \\ \mathcal{C}_i & \mathcal{D}_i \end{bmatrix}, \qquad Y_i = \begin{bmatrix} X_i \\ U_i \end{bmatrix}, \qquad J_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix},$$

and $I_n, 0_n$ are the identity and zero matrices. It is well known that the Riccati matrix difference equation

$$R_W[Q] = 0_n, \qquad i = 0, \dots, N, \tag{1.2}$$

has a symmetric solution $Q_i^\top = Q_i$ iff there exists a conjoined basis of system (1.1)

$$Y_i^\top J_{2n} Y_i = 0_n, \qquad \operatorname{rank} Y_i = n,$$

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such that the following condition holds:

$$\det X_i \neq 0, \qquad i = 0, \dots, N+1. \tag{1.3}$$

If (1.3) does not hold, we have to consider generalized solutions of equation (1.2). One possible approach to this problem which has been studied for a fairly long time (see [1] and the references therein) is connected with the technique of the Moore-Penrose inverse of matrices. For example, this technique is very popular in the modern Sturm's theory of symplectic systems. According to the definition (see [2]), a conjoined basis of (1.1) is said to have a focal point in $(i, i+1]$ if the conditions

$$\operatorname{Ker} X_{i+1} \subseteq \operatorname{Ker} X_i, \tag{1.4}$$

$$X_i X_{i+1}^{\dagger} \mathcal{B}_i \geq 0, \tag{1.5}$$

do not hold (here, $\dagger$ denotes the Moore-Penrose inverse of the matrix A, $\operatorname{Ker} A$ denotes the kernel of A, and for a symmetric matrix A we write $A \geq 0$ if A is positive semidefinite). Note (see [3]) that $\operatorname{Ker} A \subseteq \operatorname{Ker} C$ is equivalent to

$$C = CA^{\dagger}A. \tag{1.6}$$

Another necessary and sufficient condition for (1.4) is given in [4]

$$M_i = \left(I_n - X_{i+1}X_{i+1}^{\dagger}\right) \mathcal{B}_i = 0_n, \tag{1.7}$$

and a conjoined basis has a focal point of multiplicity $\operatorname{rank} M_i = m_1(i)$ in the point $i+1$ provided condition (1.4) does not hold. If a conjoined basis without focal points in $(0, N+1]$ is considered, then conditions (1.4),(1.5) are equivalent to the existence of a symmetric solution of the "implicit Riccati equation" (see [2,3])

$$R_W[Q]X_i = 0_n, \qquad i = 0, \dots N, \tag{1.8}$$

such that the following condition holds:

$$\left(\mathcal{D}_i^{\top} - \mathcal{B}_i^{\top} Q_{i+1}\right) \mathcal{B}_i \geq 0, \qquad i = 0, \dots, N. \tag{1.9}$$

If the matrix X_i is nonsingular, condition (1.9) may be rewritten as

$$(\mathcal{A}_i + \mathcal{B}_i Q_i)^{-1} \mathcal{B}_i \geq 0, \tag{1.10}$$

for the solution of the Riccati equation (1.2). The system is disconjugate on $(0, N+1]$ if the solution with initial conditions $X_0 = 0_n$, $U_n = I_n$ (the principal solution at 0) does not have focal points in $(0, N+1]$.

Another (and alternative) approach to the problem of solvability of the Riccati equation is connected with the application of transformations which change the rank of the upper block of a conjoined basis. Among these transformations there are transformations with orthogonal matrices which play an important role in numerical analysis (see, for example, [5]) and the geometrical theory of differential equations (see [6]). The major difficulty of these transformations is that they do not preserve, in general, oscillation properties of conjoined bases. In particular, they do not preserve the distribution of focal points (and the property of having no focal points) of a given conjoined basis. Therefore, the definition of the oscillation properties of the initial system in terms of the new transformed basis may not be such an easy task. From this point of view, this work develops this alternative approach to the problem of solvability of the Riccati equation. We consider the solutions of a transformed Riccati equation

$$R_{\tilde{W}}[Q_j] = 0_n, \tag{1.11}$$

where the matrices $\tilde{W}_i$ are defined by the formula $\tilde{W}_i = (\mathfrak{N}_{j(i+1)})^\top W_i \mathfrak{N}_{j(i)}$, and $\mathfrak{N}_{j(i)}$ are the symplectic orthogonal matrices uniquely determined by the values of a function $j = j(i)$, $i = 0, \dots, N+1$ (it is called an integration path for a conjoined basis Y_i (see [7])). The transformations with matrices $\mathfrak{N}_{j(i)}$ are connected with permutations of rows of a conjoined basis in such a way that condition (1.3) holds for the transformed basis for any i. The idea of this approach was offered in [8] for linear differential systems and was developed in [9] for Hamiltonian differential systems. Consider the example which illustrates the concept of an integration path.

EXAMPLE 1.1. Let the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 - 2t^2 + 4t & 2 - (1-t)^3 \\ 2 - (1-t)^3 & 1 - t^2 - t^2(1-t)^2 \end{bmatrix}, \qquad B = -I_2$$

be blocks of the matrix of the differential Hamiltonian systems

$$Y'(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & -A(t)^\top \end{bmatrix} Y(t), \qquad B(t) = B(t)^\top, \quad C(t) = C(t)^\top.$$

Consider the conjoined basis of this system for $t \in [0,3]$ with initial condition $Y(0) = [I_2 \quad 0_2]^\top$. The solution of the associated differential Riccati equation has the form

$$Q(t) \equiv Q_0(t) = \frac{1}{1-t} \begin{bmatrix} 1 - (1-t)^2 & t(1-t) \\ t(1-t) & t(1-t)^2 \end{bmatrix}.$$

Then, the point $t = 1$ is focal for the given conjoined basis (for the differential case, it means that condition (1.3) does not hold). Note that the transformed conjoined basis $Y_3(t) = J_{2n}^\top Y(t)$, $n = 2$ has the same focal point $t = 1$, and the respective solution of the transformed Riccati equation has the form

$$Q_3(t) \equiv -Q(t)^{-1} = \frac{1}{t(1-t)^2} \begin{bmatrix} -(1-t)^2 & 1-t \\ 1-t & t-2 \end{bmatrix}.$$

This situation which does not take place for the scalar Riccati equation ($n = 1$) was discussed for the matrix case in [1]. In this work the existence of other transformations such that condition (1.3) held for the new basis was considered as an open problem. Introduce the following transformations of a conjoined basis: $Y_j(t) = \mathfrak{N}_j^\top Y(t)$, where $j \in \{0,1,2,3\}$, and $\mathfrak{N}_0 = I_{2n}$, $\mathfrak{N}_3 = J_{2n}$, $n = 2$,

$$\mathfrak{N}_j = \begin{bmatrix} F_j & G_j \\ -G_j & F_j \end{bmatrix}, \qquad G_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_j = I_2 - G_j, \qquad j = 1, 2.$$

The solutions of the Riccati equations which correspond to the case $j = 1, 2$ are the matrices

$$Q_1(t) = \frac{1}{t(t-1)} \begin{bmatrix} t^2(t-1) & t \\ t & 1 \end{bmatrix}, \qquad Q_2(t) = \frac{t-1}{t(2-t)} \begin{bmatrix} 1 & t \\ t & t^2(t-1) \end{bmatrix}.$$

It is easy to see that only the basis $Y_2(t)$ does not have $t = 1$ as a focal point. Introduce the function $j = j(t) \in \{0,1,2,3\}$ such that condition (1.3) holds for $Y_j(t)$ for any t. There are different ways of choosing such functions for the given example. So, one can put $j(t) = 0$, $0 \le t \le t_1 < 1$, $j(t) = 2$, $t_1 \le t \le t_2 < 2$, $j(t) = 3$, $t_2 \le t \le 3$. This function $j(t)$ is called an integration path for the given basis $Y(t)$, and $Q_j(t)$, $j = j(t)$ may be considered as the solution of the Riccati equation along this path. Note also that there exist the principal minors of $Q_j(t)$, $j = j(t)$ whose zeros coincide with zeros of $\det X(t)$. The results of the numerical evaluation of $Q_j(t)$ along $j = j(t)$ are given in [7].

The case of (1.1) was examined in [7] where uniformly bounded along an integration path solutions of (1.11) were introduced.

In this work, we consider the special row transformations of a conjoined basis and offer a concept of a special integration path $j = j(i)$, $i = 0, \dots, N+1$ (Definition 2.6), connected with $\operatorname{rank} X_i$. This new concept gives us the possibility to avoid the evaluation of pseudoinverse matrices in (1.5)–(1.7) (see Theorem 3.6). So, it is possible to connect two factors of the skeleton factorization (see [10]) of X_i with the solutions of the transformed Riccati equation (1.11) (the first factor) and the transformed conjoined basis $\mathfrak{N}_j Y_i$ (the second factor) and then formulate the definition of a focal point (Definition 3.9), and the disconjugacy criteria (Corollary 3.8), in terms of these factors. The advantage of the Riccati equation (1.11) over equation (1.8) is that equation (1.8) is a rather complicated while equation (1.11) is much easier to deal with. So, we have the solution of (1.11) in a form which uses only one conjoined basis of (1.1) while the symmetric solution of (1.8) is given in [2,3] in terms of two conjoined bases of (1.1). Using the concept of an integration path we develop the Riccati technique for the case when condition (1.3) does not hold. To illustrate it we formulate and prove the theorems (Theorems 4.1 and 4.4) which may be viewed as analogues of Theorem 1 in [11].

THEOREM 1.2. *Let (X,U) and $(\tilde{X},\tilde{U})$ be conjoined bases of the Hamiltonian difference system $\Delta X = \mathcal{A}_i X_i + \mathcal{B}_i U_i$, $\Delta U = \mathcal{C}_i X_i - \mathcal{A}_i^\top U_i$ with $\operatorname{Im} \tilde{X}_M \subseteq \operatorname{Im} X_M$, $\tilde{X}_M^\top(\tilde{Q}_M - Q_M)\tilde{X}_M \geq 0$, $\tilde{Q} = \tilde{X}\tilde{X}^\dagger \tilde{U}\tilde{X}^\dagger$, $Q = XX^\dagger U X^\dagger$. If (X,U) has no focal points in $(M, N+1]$, then neither does $(\tilde{X},\tilde{U})$.*

2. PRELIMINARIES

We introduce the transformations for solutions of system (1.1)

$$Y_i = \mathfrak{N}_j Y_j^i, \quad \mathfrak{N}_j = \begin{bmatrix} F_j & G_j \\ -G_j & F_j \end{bmatrix}, \quad Y_j^i = \begin{bmatrix} X_j^i \\ U_j^i \end{bmatrix}, \qquad j = j(i). \tag{2.1}$$

When writing $Y_{j(i)}, X_{j(i)}, U_{j(i)}, Q_{j(i)}$, we always mean $Y_{j(i)}^i, X_{j(i)}^i, U_{j(i)}^i, Q_{j(i)}^i$. For the matrices $Y_{j(i)}$, we have the transformed system

$$Y_{j(i+1)} = \tilde{W}_i Y_{j(i)}, \qquad \tilde{W}_i = \mathfrak{N}_{j(i+1)}^\top W_i \mathfrak{N}_{j(i)} = \begin{bmatrix} \tilde{\mathcal{A}}_i & \tilde{\mathcal{B}}_i \\ \tilde{\mathcal{C}}_i & \tilde{\mathcal{D}}_i \end{bmatrix}. \tag{2.2}$$

Consider the definition of the matrices $\mathfrak{N}_j$. We say that a matrix $\mathfrak{N} \in \Omega_{\mathfrak{N}}$ if it may be written in the form (2.1) with $n \times n$ diagonal blocks F, G which obey the conditions

$$F^2 + G^2 = I_n, \qquad FG = 0_n, \tag{2.3}$$

$$F \geq 0, \qquad G \geq 0. \tag{2.4}$$

Condition (2.3) defines a group of symplectic orthogonal matrices. It is easy to verify that there exist only 2^n symplectic orthogonal matrices $\mathfrak{N}_j$ defined by (2.3),(2.4). The number $j = j(i)$ of any $\mathfrak{N}_j$ takes the values from the set $\{0, 1, \dots, 2^n - 1\}$, and the diagonal of G_j is composed of the zeros and ones that constitute the binary representations of $j = j(i)$. In this case, we have $\mathfrak{N}_0 = I_{2n}$, $\mathfrak{N}_{2^n-1} = J_{2n}$, $\mathfrak{N}_j = J_{2n}\mathfrak{N}_{2^n-1-j}^\top$.

The treatment of the set $\Omega_{\mathfrak{N}}$ is justified by the following theorem.

THEOREM 2.1. *For any conjoined basis of system (1.1) there exists a function $j = j(i)$, $i = 0, \dots, N+1$ such that the matrix $X_{j(i)}$ in (2.1) is nonsingular*

$$\det\left(X_{j(i)}\right) = \det(F_j X_i - G_j U_i) \neq 0, \qquad i = 0, \dots, N+1. \tag{2.5}$$

Hence, there exists the symmetric solution $Q_j = U_j X_j^{-1}$ of the Riccati equation (1.11) associated with the transformed system (2.2).

PROOF. See [9], where the relations between Plucker's (Grassmann's) coordinates of the Lagrangian plane in the symplectic space [6] are used. Note also that this theorem deals with the case of "symmetric" permutations of rows and becomes trivial for the case of all possible permutations of rows of Y_i because this is a full rank matrix. ∎

DEFINITION 2.2. *A function $j = j(i)$, $i = 0, \dots, N+1$ is called an integration path for a conjoined basis of system (1.1) if condition (2.5) holds. In this case, one can consider $Q^i_j = Q_j = U_j X_j^{-1}$ as the solution along the path $j = j(i)$, $i = 0, \dots, N+1$.*

It does not follow from this definition that an integration path is uniquely defined. But, if we know an integration path $j(i)$ for a conjoined basis and Q_j, we possess all information on the existence of other integration paths for given basis and $\operatorname{rank} X_i$. To show this, consider the properties of transformations (2.1). Let $l = l(i)$, $i = 0, \dots, N+1$, be another function and matrices $\mathfrak{N}_l$ with blocks F, G belong to $\Omega_{\mathfrak{N}}$. Then, we have the following connection between two transformations of the conjoined basis Y_i:

$$Y_l = \mathfrak{N}_p Y_j, \qquad \mathfrak{N}_p = \mathfrak{N}_l^\top \mathfrak{N}_j = \begin{bmatrix} F_p & G_p \\ -G_p & F_p \end{bmatrix}. \tag{2.6}$$

Certainly, if $l = 0$, (2.6) passes into (2.1) (we have $Y^i_0 \equiv Y_i$). The diagonal matrices F_p, G_p obey conditions (2.3), but generally speaking, conditions (2.4) do not hold for F_p, G_p. We can formulate the following proposition.

PROPOSITION 2.3. *Let $\mathfrak{N}_j, \mathfrak{N}_l$ belong to $\Omega_{\mathfrak{N}}$. The matrix $\mathfrak{N}_p \in \Omega_{\mathfrak{N}}$ iff $G_j \geq G_l$ ($F_l \geq F_j$). The matrix $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$ iff $G_l \geq G_j$ ($F_j \geq F_l$).*

PROOF. One can use the following representations for F_p, G_p:

$$F_p = F_l F_j + G_l G_j, \qquad G_p = F_l G_j - F_j G_l, \tag{2.7}$$

or the reverse representations

$$F_l = F_p F_j + G_p G_j, \qquad G_l = F_p G_j - F_j G_p. \tag{2.8}$$

By (2.7) and the definition of $\Omega_{\mathfrak{N}}$, we have that $F_p \geq 0$, and $G_p \geq 0$ iff $F_j G_l = 0_n$. But $G_j - G_l = F_l - F_j = G_j F_l - G_l F_j \geq 0$ iff $F_j G_l = 0_n$. In this case, $G_j - G_l = F_l - F_j = G_p \geq 0$, or

$$F_l = F_j + G_p, \qquad G_l = G_j - G_p. \tag{2.9}$$

Using (2.9), we also have that the condition $\mathfrak{N}_p \in \Omega_{\mathfrak{N}}$ is equivalent to the condition $G_j \geq G_p$ ($F_p \geq F_j$). The second claim can be proved in the same manner. ∎

Let $j = j(i)$, $i = 0, \dots, N+1$, be an integration path for a conjoined basis Y_i. Consider the formula which connects all principal minors of matrix $Q_j = U_j X_j^{-1}$ with the corresponding minors of order n of Y_i. We have

$$M[G_p Q G_p] = (-1)^{\operatorname{rank}(G_l F_j)} \frac{\det(F_l X_i - G_l U_i)}{\det X_j}, \tag{2.10}$$

where $M[G_p Q G_p]$ denotes the principal minor of $Q_j = U_j X_j^{-1}$ located in rows and columns defined by positions of nonzero elements in the diagonal of the matrix G_p. The minor $\det(F_l X_i - G_l U_i)$ of order n coincides (accurate to a sign) with some minor of Y_i. The location of this minor is uniquely determined by F_l, G_l and formulae (2.8). It follows from (2.10) that another integration path $l = l(i)$, $i = 0, \dots, N+1$ exists if and only if the principal minor $M[G_p Q G_p]$ with G_p defined by (2.7) is nonzero for any $i = 0, \dots, N+1$ or

$$\operatorname{rank}(G_p Q_j G_p) = \operatorname{rank} G_p > 0. \tag{2.11}$$

If condition (2.11) holds, one can consider the formula $Q_l = (I_n - F_p Q_j G_p)(F_p Q_j F_p - (G_p Q_j G_p)^\dagger) \times (I_n - G_p Q_j F_p)$ which connects the solutions $Q^i_{l(i)} = Q_{l(i)}$, $Q^i_{j(i)} = Q_{j(i)}$ along the different integration paths. For example, if $j(i) \equiv 0$, $l(i) \equiv 2^n - 1$, we have $Q^i_{2^n-1} = -(Q^i_0)^{-1} \equiv -Q_i^{-1}$, moreover, $Q^i_{2^n-1-j} = -(G_p Q^i_j G_p)^{-1}$, $G_p = G_j - F_j$.

Now, we introduce a special integration path for a conjoined basis.

LEMMA 2.4. *Let the matrices $\mathfrak{N}_j \in \Omega_{\mathfrak{N}}$, $j = j(i)$, $i = 0, \dots, N+1$, and*

$$\operatorname{rank} X_i = \operatorname{rank} F_j X_i = \operatorname{rank} F_j, \tag{2.12}$$

where X_i is the upper block of a conjoined basis Y_i. Then, $j = j(i)$, $i = 0, \dots, N+1$ is the integration path for Y_i, and the solution along this path Q_j satisfies the condition

$$G_j Q_j G_j = 0_n. \tag{2.13}$$

PROOF. Let condition (2.12) hold. We have to prove that (2.5) holds. As is well known, if we multiply Y_i by a nondegenerate $n \times n$ matrix M, Plucker's coordinates of Y_i will multiply by $\det M$. Hence, if (2.5) does not hold, we can consider the new conjoined basis $\tilde{Y}_i = Y_i M$, $\det M \neq 0$ which has zero columns (see also [6, p. 225]), and we arrive at contradiction with the definition of Y_i. Then, the path defined by (2.12) does exist. To prove (2.13), consider the factorization of X_i,

$$X_i = (F_j X_j + G_j U_j) = (F_j + G_j Q_j) X_j = (I_n + G_j Q_j F_j)(F_j + G_j Q_j G_j) X_j, \tag{2.14}$$

where $j = j(i)$ is an integration path. Since the matrices $I_n + G_j Q_j F_j$, X_j are nonsingular, we have that $\operatorname{rank} X_i = \operatorname{rank}(F_j + G_j Q_j G_j) = \operatorname{rank} F_j + \operatorname{rank}(G_j Q_j G_j)$ and $\operatorname{rank} X_i = \operatorname{rank} F_j$ iff $G_j Q_j G_j = 0_n$. ∎

REMARK 2.5. Note that if we have a solution Q_j along an integration path, we can construct another path defined by (2.12). So, if in (2.14) $\operatorname{rank}(G_j Q_j G_j) > 0$, one can find the principal minor $M[G_p Q G_p]$ of Q_j such that $G_j \geq G_p \geq 0$, $\operatorname{rank}(G_j Q_j G_j) = \operatorname{rank}(G_p Q_j G_p) = \operatorname{rank} G_p > 0$. Hence, we can introduce the new integration path $l = l(i)$ determined by (2.9) with $\operatorname{rank} X_i = \operatorname{rank}(F_l X_i) = \operatorname{rank} F_l$, and $G_l Q_l G_l = 0_n$.

DEFINITION 2.6. *The functions $j = j(i)$, $i = 0, \dots, N+1$, defined by condition (2.12) are called a special path for a conjoined basis Y_i.*

3. MAIN RESULTS

LEMMA 3.1. *Let $j = j(i)$, $i = 0, \dots, N+1$, be a special integration path for a conjoined basis Y_i. Then, we have $\operatorname{Im} X_i = \operatorname{Im}(F_j + G_j Q_j F_j)$, $\operatorname{Ker} X_i = \operatorname{Ker}(F_j X_j)$.*

PROOF. It follows from the factorization (2.14) of X_i, which for case (2.13) may be rewritten as

$$X_i = (F_j + G_j Q_j F_j)(F_j X_j) = AB, \tag{3.1}$$

that (3.1) is the analog of the skeleton factorization [10] of X_i such that $\operatorname{rank} X_i = \operatorname{rank} A = \operatorname{rank} B = \operatorname{rank} F_j$, $A = AF_j$, $B = F_j B$, $BB^\dagger = F_j$, $A^\dagger A = F_j$, $X_i^\dagger = B^\dagger A^\dagger$, so that $X_i B^\dagger = A$, $A^\dagger X_i = B$. Hence, by the last conditions and (3.1), we have that $\operatorname{Im} X_i = \operatorname{Im} A$, $\operatorname{Ker} X_i = \operatorname{Ker} B$. ∎

Using (3.1), one can get the following proposition.

PROPOSITION 3.2. *Let $j = j(i)$, $i = 0, \dots, N+1$, be a special integration path for a conjoined basis Y_i. For any $n \times n$ matrix C the following conditions are equivalent:*

$$\operatorname{Ker} X_{i+1} \subseteq \operatorname{Ker} C \Leftrightarrow C X_{j(i+1)}^{-1} = C X_{j(i+1)}^{-1} F_{j(i+1)}, \tag{3.2}$$

$$\begin{aligned} \operatorname{Im} C \subseteq \operatorname{Im} X_i &\Leftrightarrow C^\top \left(I_n - F_{j(i)} Q_{j(i)} G_{j(i)}\right) = C^\top F_{j(i)} \\ &\Leftrightarrow C^\top = C^\top \left(F_{j(i)} + Q_{j(i)} G_{j(i)}\right). \end{aligned} \tag{3.3}$$

If $C^\top = C$, we have in (3.3) $C = (F_j + G_j Q_j) C (F_j + Q_j G_j)$, $j = j(i)$.

PROOF. One direction of (3.2) is trivial because it follows from the right-hand side of (3.2) that $C = C X_{j(i+1)}^{-1} F_{j(i+1)} X_{j(i+1)}$, and by Lemma 3.1, $\operatorname{Ker} X_{i+1} = \operatorname{Ker}(F_{j(i+1)} X_{j(i+1)})$. If $\operatorname{Ker} X_{i+1} \subseteq$

$\operatorname{Ker} C$, then, by (1.6), we have $C = C(F_jX_j)^\dagger F_jX_j$, $j = j(i+1)$ or $CX_j^{-1} = C(F_jX_j)^\dagger F_j$, $j = j(i+1)$. Then, for $j = j(i+1)$ it follows that

$$CX_j^{-1} = C(F_jX_j)^\dagger = CX_j^{-1}F_j. \tag{3.4}$$

Equivalence (3.3) can be proved in the same manner. ∎

Consider Wronski's identity for two solutions $\hat{Y}_i$ and Y_i of system (1.1),

$$w_i = \hat{Y}_i^\top J_{2n} Y_i = \text{const}. \tag{3.5}$$

LEMMA 3.3. *Let $j = j(i)$, $i = 0, \dots, N+1$, be a special integration path for a conjoined basis Y_i, and $\hat{Y}_i = \begin{bmatrix} \hat{X}_i \\ \hat{U}_i \end{bmatrix}$ is the solution of (1.1). Then,*

$$\operatorname{Im} \hat{X}_i \subseteq \operatorname{Im} X_i \Leftrightarrow w_i X_{j(i)}^{-1} = w_i X_{j(i)}^{-1} F_{j(i)}, \tag{3.6}$$

where $X_{j(i)}$ is the upper block of the transformed conjoined basis $Y_{j(i)}$ (2.1).

PROOF. It is easy to verify that

$$w_i X_j^{-1} = -\hat{U}_i^\top (F_j + G_jQ_j) + \hat{X}_i^\top(-G_j + F_jQ_jG_j) + \hat{X}_i^\top F_jQ_jF_j.$$

We have $G_jQ_j \equiv G_jQ_jF_j$ along the special path, then $w_iX_{j(i)}^{-1} = w_iX_{j(i)}^{-1}F_{j(i)}$ iff the second addend equals zero, or $\hat{X}_i^\top = \hat{X}_i^\top(F_{j(i)} + Q_{j(i)}G_{j(i)})$. The last condition is equivalent to $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ because of Proposition 3.2 with $C = \hat{X}_i$. ∎

REMARK 3.4. Using (3.2),(3.6),(3.5), we have that $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ is equivalent to $\operatorname{Ker} X_i \subseteq \operatorname{Ker} w_i = \operatorname{Ker} w_{i+1}$. Then, $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$, $\operatorname{Ker} X_{i+1} \subseteq \operatorname{Ker} X_i$ imply $\operatorname{Ker} X_{i+1} \subseteq \operatorname{Ker} w_{i+1}$ or $\operatorname{Im}\hat{X}_{i+1} \subseteq \operatorname{Im} X_{i+1}$ (see also [2, Remark 1(v)]).

Consider the matrix w_i. It is easy to see that the matrix

$$w_iX_i^{-1}\hat{X}_i = \hat{X}_i^\top Q_i\hat{X}_i - \hat{U}_i^\top\hat{X}_i \tag{3.7}$$

is symmetric if $\hat{Y}_i$ is a conjoined basis as well, and X_i is nonsingular (in this case $Q_i \equiv Q_j^i$, $j(i) \equiv 0$ is a solution of the Riccati equation (1.2)). If X_i is singular, by [11], we have that $w_iX_i^\dagger\hat{X}_i$ is symmetric if the condition $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ holds. We offer to consider

$$w_i = \hat{Y}_i^\top J_{2n} Y_i = \hat{Y}_i^\top \mathfrak{N}_j J_{2n} \mathfrak{N}_j^\top Y_i = \hat{Y}_j^\top J_{2n} Y_j, \qquad j = j(i),$$

where $j = j(i)$ is an integration path for a conjoined basis $Y_i, Y_j = \mathfrak{N}_j^\top Y_i$,

$$\hat{Y}_j = \mathfrak{N}_j^\top \hat{Y}_i = \begin{bmatrix} \hat{X}_j \\ \hat{U}_j \end{bmatrix}, \tag{3.8}$$

and $\mathfrak{N}_j \in \Omega_{\mathfrak{N}}$. Certainly, the matrix $w_iX_j^{-1}\hat{X}_j$ is symmetric and equals the right-hand side of (3.7), where the index i is replaced by $j(i)$, and Q_j, $j = j(i)$ is the solution along the path $j = j(i)$, $i = 0, \dots, N+1$.

LEMMA 3.5. *Let $j = j(i)$, $i = 0, \dots, N+1$, be a special integration path for a conjoined basis $Y_i, \hat{Y}_i$ is another conjoined basis, and $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ holds. Then, $w_iX_j^{-1}\hat{X}_j = w_iX_j^{-1}\hat{X}_i = w_iX_i^\dagger\hat{X}_i$, where $\hat{X}_j$ is given by (3.8).*

PROOF. By Lemma 3.3, using $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ and (3.4) we have that $w_iX_j^{-1} = w_iX_j^{-1}F_j = w_i(F_jX_i)^\dagger$. Then, $\hat{X}_j = F_j\hat{X}_i - G_j\hat{U}_i$ yields $w_iX_j^{-1}\hat{X}_j = w_iX_j^{-1}\hat{X}_i = w_i(F_jX_i)^\dagger F_j(F_j\hat{X}_i - G_j\hat{U}_i) = w_i(F_jX_i)^\dagger F_j\hat{X}_i = w_iB^\dagger A^\dagger A\hat{X}_i = w_iX_i^\dagger A\hat{X}_i$, where the matrices A, B are the factors in the skeleton factorization (3.1) of X_i. Note that $A\hat{X}_i = \hat{X}_i$, $A = F_j + G_jQ_j$ because of $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$ and Proposition 3.2. ∎

Now, it is possible to prove the main result.

THEOREM 3.6. *Let $j = j(i)$, $i = 0, \ldots, N+1$, be a special integration path for a conjoined basis Y_i, and Q_j is the solution of (1.11) along this path. Condition (1.4) is equivalent to the condition*

$$F_{j(i)} P_{j(i)} G_{j(i+1)} = 0_n, \qquad P_{j(i)} = X_{j(i)} X_{j(i+1)}^{-1} = \left(\tilde{\mathcal{A}}_i + \tilde{\mathcal{B}}_i Q_{j(i)}\right)^{-1}, \tag{3.9}$$

and if (1.4) (or (3.9)) holds, then (1.5) is equivalent to

$$F_{j(i)} D_{j(i)} F_{j(i)} \geq 0, \tag{3.10}$$

with

$$D_{j(i)} = P_{j(i)} \tilde{\mathcal{B}}_i. \tag{3.11}$$

PROOF. Put $C = F_{j(i)} X_{j(i)}$ in equivalence (3.2). Hence, we have $F_{j(i)} X_{j(i)} X_{j(i+1)}^{-1} = F_{j(i)} \times X_{j(i)} X_{j(i+1)}^{-1} F_{j(i+1)}$, where $X_{j(i)} X_{j(i+1)}^{-1} = P_{j(i)}$. Then, conditions (1.4) and (3.9) are equivalent.

Consider Wronski's identity $w_{i+1} = w_i$, Lemmas 3.3 and 3.5 for the particular case $\hat{Y}_i = [0_n, I_n]^\top$. We have $\hat{X}_{i+1} = \mathcal{B}_i$, $\hat{U}_{i+1} = \mathcal{D}_i$, and $-w_{i+1} X_{j(i+1)}^{-1} = -w_i X_{j(i+1)}^{-1} = X_i X_{j(i+1)}^{-1} = (I_n + G_{j(i)} Q_{j(i)}) F_{j(i)} P_{j(i)}$, where we use (3.1). By Lemma 3.3, $\operatorname{Im} \mathcal{B}_i \subseteq \operatorname{Im} X_{i+1}$ is equivalent to $w_{i+1} X_{j(i+1)}^{-1} G_{j(i+1)} = 0_n$, then the condition $\operatorname{Im} \mathcal{B}_i \subseteq \operatorname{Im} X_{i+1}$ is equivalent to (3.9). Now, consider $-w_{i+1} X_{i+1}^\dagger \mathcal{B}_i = -w_i X_{i+1}^\dagger \mathcal{B}_i = X_i X_{i+1}^\dagger \mathcal{B}_i$. This matrix is symmetric, since by $\operatorname{Im} \mathcal{B}_i \subseteq \operatorname{Im} X_{i+1}$ and Lemma 3.5 it coincides with the symmetric matrix $C = -w_{i+1} X_{j(i+1)}^{-1} \hat{X}_{j(i+1)} = (F_{j(i)} + G_{j(i)} Q_{j(i)}) P_{j(i)} \mathcal{B}_i$, where $\hat{X}_{j(i+1)}$ is the upper block of $\hat{Y}_{j(i+1)}$. Moreover, because of $\operatorname{Im} w_{i+1} X_{i+1}^\dagger \mathcal{B}_i \subseteq \operatorname{Im} X_i$, we have by Proposition 3.2 that $C = C^\top = (F_j + G_j Q_j F_j) C (F_j + F_j Q_j G_j)$, $j = j(i)$. Using the identity $F_{j(i+1)} \tilde{\mathcal{B}}_i F_{j(i)} = F_{j(i+1)} \mathcal{B}_i F_{j(i)}$ we have that C also coincides with

$$\left(F_{j(i)} + G_{j(i)} Q_{j(i)} F_{j(i)}\right) D_{j(i)} \left(F_{j(i)} + F_{j(i)} Q_{j(i)} G_{j(i)}\right),$$

for D_j given by (3.11). Then, $C \geq 0$ is equivalent to $F_{j(i)} D_{j(i)} F_{j(i)} \geq 0$ because the matrix $I_n + G_j Q_j F_j$ is nonsingular. ∎

REMARK 3.7.

(i) Note that if (1.3) holds for $i = 0, \ldots, N+1$, then $F_j \equiv I_n$, $G_j \equiv 0_n$, $j = j(i)$, $i = 0, \ldots, N+1$, hence, the integration path defined by the matrix G_j is trivial ($j(i) \equiv 0$), and the solution along this path Q_0^i coincides with the classical solution of (1.2). In this case, condition (3.9) is trivial, and (3.10) passes into (1.10).

(ii) Note that (3.9) certainly implies $\operatorname{rank} F_{j(i+1)} \geq \operatorname{rank} F_{j(i)}$ because it is impossible for the nonsingular matrix $(\tilde{\mathcal{A}}_i + \tilde{\mathcal{B}}_i Q_{j(i)})^{-1}$ to have a zero block of arbitrary dimensions. If $F_{j(i)} = I_n$ for $i = i_0$, then $F_{j(i)} = I_n$ for $i \geq i_0$, and $j(i) = 0$, $i \geq i_0$.

(iii) Certainly, condition (3.9) is equivalent to (1.7) as well. Moreover, $\operatorname{rank} M_i = \operatorname{rank}(F_{j(i)} \times P_{j(i)} G_{j(i+1)}) = m_1(i)$ provided condition (3.9) does not hold (see Section 1).

COROLLARY 3.8. *Let Y_i be the principal solution at 0. Then, system (1.1) is disconjugate iff conditions (3.9),(3.10) hold for the solution of equation (1.11) with the initial conditions $Q_{j(0)}^0 = 0_n$, $j(0) = 2^n - 1$.*

Now, it is possible to formulate the following definition of a focal point.

DEFINITION 3.9. *Let $j = j(i)$, $i = 0, \ldots, N+1$, be a special integration path for a conjoined basis Y_i, and $Q_{j(i)}$ is the solution along this path. Then, Y_i has a focal point in $(i, i+1]$ if conditions (3.9),(3.10) do not hold.*

4. SEPARATION THEOREMS

In this section, we formulate separation results which are similar to Theorem 1.2 (see Section 1) in terms of solutions of (1.11). We also give the new proof of these results which uses only Wronski's identity and equation (1.11).

Let $l = l(i)$, $j = j(i)$, $i = 0, \dots, N+1$, be integration paths for two conjoined bases $\hat{Y}_i, Y_i$, respectively. Rewrite Wronski's identity $w_{i+1} = w_i$ in the form

$$\Delta_{i+1} = \hat{P}_{l(i)}^{\top} \Delta_i P_{j(i)}, \tag{4.1}$$

for

$$\Delta_i = \hat{X}_l^{-1\top} w_i X_j^{-1} = \begin{bmatrix} I_n \\ \hat{Q}_l \end{bmatrix}^{\top} \mathfrak{N}_p J_{2n} \begin{bmatrix} I_n \\ Q_j \end{bmatrix},$$

$$\hat{P}_l = \left(\hat{\mathcal{A}}_i + \hat{\mathcal{B}}_i \hat{Q}_{l(i)}\right)^{-1}, \qquad P_j = \left(\tilde{\mathcal{A}}_i + \tilde{\mathcal{B}}_i Q_{j(i)}\right)^{-1},$$

$l = l(i)$, $j = j(i)$, $p = p(i)$, the matrix $\mathfrak{N}_p = \mathfrak{N}_l^{\top} \mathfrak{N}_j$ connects the matrices of transformations (2.1), $\hat{Y}_i = \mathfrak{N}_l \hat{Y}_l$, and $\hat{Q}_l = \hat{Q}_l^i$, $Q_j = Q_j^i$ are solutions of two transformed Riccati equations with coefficients which are blocks of $\hat{W}_i, \tilde{W}_i$, respectively. It is important to point out that $\Delta_i = Q_j - \hat{Q}_l$ if $\mathfrak{N}_p = I_{2n}$, in this case $l(i) \equiv j(i)$ and $\Delta_i = \Delta_i^{\top}$. If $\mathfrak{N}_p \neq I_{2n}$, we may introduce the symmetric matrix $\tilde{\Delta}_i = \Delta_i(F_p - G_p \hat{Q}_l)$ which corresponds to the matrix $w_i X_j^{-1} \hat{X}_j$ considered in Lemma 3.5. Note that all results (see Section 3) connected with w_i may be proved for Δ_i as well. Thus, if $l = l(i)$, $j = j(i)$, $i = 0, \dots, N+1$, are special integration paths, then $\operatorname{Im} \hat{X}_i \subseteq \operatorname{Im} X_i \Leftrightarrow \Delta_i = \Delta_i F_{j(i)}$ and also $\operatorname{Im} X_i \subseteq \operatorname{Im} \hat{X}_i \Leftrightarrow \Delta_i = F_{l(i)} \Delta_i$. Using (4.1) one can prove the following result.

THEOREM 4.1. *Let $l = l(i)$, $j = j(i)$, $i = 0, \dots, N+1$, be integration paths for two conjoined bases $\hat{Y}_i, Y_i$, respectively, and $\hat{Q}_l, Q_j$ be solutions of two transformed Riccati equations along these paths. Let $D_{j(i)} \geq 0$, $i = 0, \dots, N$ for D_j given by (3.11). Then, the condition $\tilde{\Delta}_i \leq 0$ for $i = 0$ implies the same condition for any $i = 0, \dots, N+1$. The integration path $l = l(i)$, $i = 0, \dots, N+1$ for $\hat{Y}_i$ may be chosen with F_l, G_l defined by (2.8), where F_p, G_p, $p = p(i)$ satisfy (2.3) and*

$$\operatorname{rank}\left(\hat{X}_j\right) = \operatorname{rank}\left(F_p \hat{X}_j\right) = \operatorname{rank} F_p, \tag{4.2}$$

for $\hat{X}_j$ given by (3.8). The solution $\hat{Q}_l$ along this path satisfies the conditions

$$G_{p(i)} \hat{Q}_{l(i)} G_{p(i)} = 0_n, \qquad i = 0, \dots, N+1, \tag{4.3}$$

$$F_{p(i)} \hat{P}_{l(i)} G_{p(i+1)} = 0_n, \quad F_{p(i)} \hat{P}_{l(i)} \hat{\mathcal{B}}_i F_{p(i)} \geq 0, \qquad i = 0, \dots, N. \tag{4.4}$$

PROOF. Introduce the symplectic matrix

$$\Lambda_i = \begin{bmatrix} I_n & 0_n \\ \hat{Q}_l & I_n \end{bmatrix}^{\top} \mathfrak{N}_p J_{2n} \begin{bmatrix} I_n & 0_n \\ Q_j & I_n \end{bmatrix} = \begin{bmatrix} \Delta_i & -\hat{Q}_l G_p + F_p \\ -F_p - G_p Q_j & -G_p \end{bmatrix}, \tag{4.5}$$

where $l = l(i)$, $j = j(i)$, $p = p(i)$, $i = 0, \dots, N+1$. We have the following identity:

$$\Lambda_i = \hat{A}_{l(i)}^{\top} \Lambda_{i+1} A_{j(i)}, \qquad \hat{A}_l = \begin{bmatrix} \hat{P}_{l(i)}^{-1} & \hat{\mathcal{B}}_i \\ 0_n & \hat{P}_{l(i)}^{\top} \end{bmatrix}, \qquad A_j = \begin{bmatrix} P_{j(i)}^{-1} & \tilde{\mathcal{B}}_i \\ 0_n & P_{j(i)}^{\top} \end{bmatrix}. \tag{4.6}$$

If Λ_i given by (4.5) is separated into blocks Λ_{mk}^i, $m, k = 1, 2$, using (4.6) for $\tilde{\Delta}_i = \Delta_i(F_p - G_p \hat{Q}_l) = \Lambda_{11}^i \Lambda_{12}^{i\top}$, we have that

$$\tilde{\Delta}_{i+1} = \hat{P}_{l(i)}^{\top} \left(\tilde{\Delta}_i - \Delta_i D_{j(i)} \Delta_i^{\top}\right) \hat{P}_{l(i)}, \tag{4.7}$$

for D_j given by (3.11). Then, $\tilde{\Delta}_i \leq 0$ implies $\tilde{\Delta}_{i+1} \leq 0$. If an integration path $l = l(i)$, $i = 0, \dots, N+1$ is chosen with F_l, G_l defined by (2.8) for F_p, G_p given by (4.2),(2.3), one can possibly reword (without using condition (2.4)) the proof of Lemma 2.4 for the transformed basis $\hat{Y}_j$ (see (3.8)), and hence, the matrix $F_p \hat{X}_j + G_p \hat{U}_j = F_l \hat{X}_i - G_l \hat{U}_i$ is nonsingular and (4.3) holds. Then, by Theorem 3.6, conditions (4.4) imply that $\hat{Y}_j$ does not have focal points in $(0, N+1]$. Using (4.6) for Λ_{12}^i, we have

$$-\hat{Q}_{l(i)} G_{p(i)} + F_{p(i)} = \Delta_i D_{j(i)} + \hat{P}_{l(i)}^{-1\top} \left(-\hat{Q}_{l(i+1)} G_{p(i+1)} + F_{p(i+1)} \right) P_{j(i)}^{\top}, \tag{4.8}$$

or

$$\left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} = P_{j(i)} \left(F_{p(i+1)} - G_{p(i+1)} \hat{Q}_{l(i+1)} \right) + D_{j(i)} \Delta_i^{\top} \hat{P}_{l(i)}. \tag{4.9}$$

Then, $(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)}) \hat{P}_{l(i)} G_{p(i+1)} = 0_n$ iff $D_{j(i)} \Delta_i^{\top} \hat{P}_{l(i)} G_{p(i+1)} = 0_n$, but the last condition holds because we have $G_{p(i+1)} \tilde{\Delta}_{i+1} G_{p(i+1)} = 0_n$ ($\tilde{\Delta}_i$ is symmetric and $\tilde{\Delta}_i = \tilde{\Delta}_i F_{p(i)}$, $i = 0, \dots, N+1$). Then, by (4.7),

$$\begin{gathered} G_{p(i+1)} \hat{P}_{l(i)}^{\top} \tilde{\Delta}_i \hat{P}_{l(i)} G_{p(i+1)} = 0_n, \\ G_{p(i+1)} \hat{P}_{l(i)}^{\top} \left(-\Delta_i D_{j(i)} \Delta_i^{\top} \right) \hat{P}_{l(i)} G_{p(i+1)} = 0_n, \end{gathered}$$

and the last condition and $D_{j(i)} \geq 0$ imply $D_{j(i)} \Delta_i^{\top} \hat{P}_{l(i)} G_{p(i+1)} = 0_n$ or the first condition in (4.4), since

$$\left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} G_{p(i+1)} = \left(I_n - G_{p(i)} \hat{Q}_{l(i)} F_{p(i)} \right) F_{p(i)} \hat{P}_{l(i)} G_{p(i+1)},$$

and the first factor is nonsingular. Now, one can apply that the matrix

$$C = \left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} \tilde{\mathcal{B}}_i$$

is symmetric (we reword the proof of Theorem 3.6 for the solutions of system (2.2)). Hence, by Proposition 3.2, we have that

$$\begin{aligned} C &= \left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} \tilde{\mathcal{B}}_i \left(F_{p(i)} - \hat{Q}_{l(i)} G_{p(i)} \right) \\ &= \left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} \hat{\mathcal{B}}_i \left(F_{p(i)} - \hat{Q}_{l(i)} G_{p(i)} \right), \end{aligned}$$

where we use the identity $F_{p(i+1)} \tilde{\mathcal{B}}_i F_{p(i)} = F_{p(i+1)} \hat{\mathcal{B}}_i F_{p(i)}$. But, C may be also rewritten as

$$\begin{aligned} &C = L D_{j(i)} = L D_{j(i)} \left(F_{p(i)} - \hat{Q}_{l(i)} G_{p(i)} \right) = L \left(D_{j(i)} - D_{j(i)} \tilde{\Delta}_i D_{j(i)} \right) L^{\top}, \\ &L = \left(F_{p(i)} - G_{p(i)} \hat{Q}_{l(i)} \right) \hat{P}_{l(i)} P_{j(i)}^{-1}, \end{aligned}$$

where we apply (4.6) for Λ_{22}^i. Thus, we have the condition $C \geq 0$ which is equivalent to the last condition (4.4). ∎

REMARK 4.2. It follows from the proof of Theorem 4.1 that if for given $i = i_0$ the matrix $F_{p(i)} = I_n$, then the last condition holds for $i \geq i_0$, and then $j(i) \equiv l(i)$, $\tilde{W}_i \equiv \hat{W}_i$, $i \geq i_0$ (see also Remark 3.7(ii)). In this case the existence of $\hat{Q}_{j(i)}$, $i = i_0$ implies the existence $\hat{Q}_{j(i)}$, $i \geq i_0$.

Certainly, Theorem 4.1 deals with the case when the transformed conjoined basis Y_j does not have focal points in $(0, N+1]$ along an integration path $j = j(i)$. Now, we consider the case when the initial conjoined basis Y_i has no focal points in $(0, N+1]$. We will use the following proposition.

PROPOSITION 4.3. *Let* $\operatorname{Im}\hat{X}_i \subseteq \operatorname{Im} X_i$, $i = 0$, *and condition (3.9) holds for* Y_i, $i = 0, 1, \ldots, N$. *Then, the special path* $l = l(i)$, $i = 0, \ldots, N+1$ *for* $\hat{Y}_i$ *may be chosen with* $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$ $(F_{j(i)} \geq F_{l(i)})$, $p = p(i)$. *The condition* $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$ *is equivalent to (4.3) if* $l = l(i)$ *is a special path.*

PROOF. By Proposition 3.2 and Remark 3.4 we have $\hat{X}_l = (F_j + G_j Q_j)\hat{X}_l = (I_n + G_j Q_j F_j) F_j \hat{X}_l$, $l = l(i)$, $i = 0, \ldots, N+1$, then $\operatorname{rank}\hat{X}_i = \operatorname{rank} F_j \hat{X}_i$, and the matrix $F_{l(i)}$ which defines the basis of rows of $\hat{X}_i$ may be chosen such that $F_{j(i)} \geq F_{l(i)}$. By Proposition 2.3, we have $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$, in this case $F_l G_j \equiv 0_n$, and by (2.7), $G_p = -F_j G_l$, then (4.3) holds because $l = l(i)$ is a special path (see Lemma 2.4). Let condition (4.3) hold. Then, by Lemmas 3.1 and 3.3, using (2.7), we have

$$F_l + G_l\hat{Q}_l = (F_j + G_j Q_j)\left(F_l + G_l\hat{Q}_l\right) = (I_n + G_j Q_j F_j) F_j \left(F_p - G_p\hat{Q}_l\right) F_p, \tag{4.10}$$

then $(F_l + G_l\hat{Q}_l)G_p = (I_n + G_l\hat{Q}_l F_l) F_l G_j = 0_n$, hence, $F_l G_j \equiv 0_n$. ∎

THEOREM 4.4. *Let* $l = l(i)$, $j = j(i)$, $i = 0, \ldots, N+1$, *be special integration paths for two conjoined bases* $\hat{Y}_i, Y_i$, *respectively, and* $\hat{Q}_l, Q_j$ *are solutions of two transformed Riccati equations along these paths. Let conditions (3.9),(3.10) hold for* Q_j. *Then, the conditions* $\Delta_i = \Delta_i F_{j(i)}$, $\tilde{\Delta}_i \leq 0$ *for* $i = 0$ *imply the same conditions for any* $i = 0, 1, \ldots, N+1$. *The special path* $l = l(i)$, $i = 0, \ldots, N+1$ *for* $\hat{Y}_i$ *may be chosen with* $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$ $(F_{j(i)} \geq F_{l(i)})$, *and the solution* $\hat{Q}_l = \hat{Q}_l^i$ *along this path satisfies the conditions*

$$F_{l(i)}\hat{P}_{l(i)}G_{l(i+1)} = 0_n, \quad F_{l(i)}\hat{P}_{l(i)}\hat{\mathcal{B}}_i F_{l(i)} \geq 0, \qquad i = 0, \ldots, N. \tag{4.11}$$

PROOF. If $\Delta_i = \Delta_i F_{j(i)}$, then, by (4.1),(3.9), we have $\Delta_{i+1} = \Delta_{i+1}F_{j(i+1)}$. If (3.10) holds, then $\Delta_i D_j \Delta_i^\top = \Delta_i F_j D_j F_j \Delta_i^\top \geq 0$, and $\tilde{\Delta}_i \leq 0$ implies $\tilde{\Delta}_{i+1} \leq 0$. Note that $\tilde{\Delta}_i = \Delta_i(F_p - G_p\hat{Q}_l) = \Delta_i F_j (F_p - G_p\hat{Q}_l) = \Delta_i(F_l + G_l\hat{Q}_l)$, and $\operatorname{Im}\hat{X}_i = \operatorname{Im}(F_l + G_l\hat{Q}_l)$ by Lemma 3.1, where $\hat{X}_i$ is the upper block of $\hat{Y}_i$. If a special path $l = l(i)$ is chosen, by Proposition 4.3, with $\mathfrak{N}_p^\top \in \Omega_{\mathfrak{N}}$, then one can repeat the proof of Theorem 4.1. Thus, we have $G_{l(i+1)}\tilde{\Delta}_{i+1}G_{l(i+1)} = 0_n$, then $F_{j(i)}D_{j(i)}F_{j(i)}\Delta_i^\top\hat{P}_{l(i)}G_{l(i+1)} = 0_n$. Multiplying (4.9) by $F_{j(i)} + G_{j(i)}Q_{j(i)}$ and using (4.10),(3.9), we have instead (4.9)

$$\begin{aligned}\left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)} = (F_{j(i)} + G_{j(i)}Q_{j(i)}) \Big[P_{j(i)}\left(F_{l(i+1)} + G_{l(i+1)}\hat{Q}_{l(i+1)}\right) \\ + F_{j(i)}D_{j(i)}F_{j(i)}\Delta_i^\top\hat{P}_{l(i)}\Big].\end{aligned}$$

Then, the first condition (4.11) holds.

To prove the second condition (4.10), we will use

$$\left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}P_{j(i)}^{-1} = \left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}P_{j(i)}^{-1}F_{j(i)} = L, \tag{4.12}$$

which means that $\operatorname{Ker}(F_{j(i)}P_{j(i)}) \subseteq \operatorname{Ker}(F_{l(i)}\hat{P}_{l(i)})$ (see Proposition 3.2). Condition (4.12) follows from (4.6) and Proposition 4.3. Thus, we have

$$\begin{aligned} -\left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\Lambda_{21}^i &= F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)} \\ &= -\left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}\hat{\mathcal{B}}_i\Delta_i + \left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}P_{j(i)}^{-1},\end{aligned}$$

then (4.12) follows from $\Delta_i = \Delta_i F_{j(i)}$ and $F_l G_j \equiv 0_n$. Using (4.12), one can repeat the proof of Theorem 4.1 for the symmetric matrix

$$C = \left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}\mathcal{B}_i = \left(F_{l(i)} + G_{l(i)}\hat{Q}_{l(i)}\right)\hat{P}_{l(i)}\hat{\mathcal{B}}_i\left(F_{l(i)} + \hat{Q}_{l(i)}G_{l(i)}\right).$$

We have $C = L(D_{j(i)} - D_{j(i)}\tilde{\Delta}_i D_{j(i)})L^{\top}$, where L is given by (4.12). Then, $C \geq 0$, and the second condition (4.11) holds. ∎

Under the assumptions of Theorem 4.4, consider the matrix $\Delta_i = \Delta_i F_{j(i)}$. It is evident that $\operatorname{Im} X_i = \operatorname{Im} \hat{X}_i \Leftrightarrow G_{l(i)}\Delta_i = 0_n$. By (4.1) and Theorem 4.4 we have

$$G_{l(i+1)}\Delta_{i+1} = G_{l(i+1)}\hat{P}^{\top}_{l(i)} G_{l(i)} \Delta_i P_{j(i)},$$

then, the condition $G_{l(i)}\Delta_i = 0_n$ for $i = i_0$ implies the same condition for $i \geq i_0$. Hence, we have the particular case of Theorem 4.4 for two solutions of the transformed Riccati equation along the path $j = j(i)$, $i = 0, \dots, N+1$.

COROLLARY 4.5. *Let all assumptions of Theorem 4.4 hold. If, in addition, $\Delta_i = F_{l(i)}\Delta_i F_{j(i)}$ for $i = 0$, then $\hat{Q}_l = \hat{Q}^i_l$ exists along $l(i) \equiv j(i)$, $i = 0, \dots, N+1$, and $\Delta_i = Q_{j(i)} - \hat{Q}_{j(i)} = F_{j(i)}\Delta_i F_{j(i)}$, $\Delta_i \leq 0$ hold for $i \geq 0$. Conditions (4.11) hold for $l(i) \equiv j(i)$, $i = 0, \dots, N+1$.*

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